Name of College – S.S. College J. Bad.

Deptt – Mathematics

Topic – Inverse circular functions of complex quantities

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Time – 10 A.M to 11.30 A.M

Class – B.Sc. I (HONS)

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Study Material :→

If $\sin(x+iy) = \alpha + i\beta \Rightarrow \sin^{-1}(\alpha+i\beta) = x+iy$

Also $\sin[n\pi + (-1)^n(x+iy)] = \sin(x+iy) = \alpha+i\beta$.

$\Rightarrow \sin^{-1}(\alpha+i\beta) = n\pi + (-1)^n(x+iy)$

Similarly. If. $\cos(x+iy) = \alpha+i\beta \Rightarrow \cos^{-1}(\alpha+i\beta) = x+iy$.

Also. $\cos[2n\pi \pm (x+iy)] = \cos(x+iy) = \alpha+i\beta$

$\Rightarrow \cos^{-1}(\alpha+i\beta) = 2n\pi \pm (x+iy)$

If $\tan(x+iy) = \alpha+i\beta \Rightarrow \tan^{-1}(\alpha+i\beta) = x+iy$

Also $\tan[n\pi + (x+iy)] = \tan(x+iy) = \alpha+i\beta$

$\Rightarrow \tan^{-1}(\alpha+i\beta) = n\pi + (x+iy)$

$$\left[\begin{array}{l} \sin\theta = \sin\alpha \Rightarrow \theta = n\pi + (-1)^n\alpha \\ \cos\theta = \cos\alpha \Rightarrow \theta = 2n\pi \pm \alpha \\ \tan\theta = \tan\alpha \Rightarrow \theta = n\pi + \alpha \end{array}\right]$$

Remarks :→ ∵ Inverse circular functions of a complex quantity is many-valued function.

* Principal value of inverse Function is obtained by putting $n=0$ in the General value of the Function.

Thus General value of

$$\sin^{-1}(\alpha+i\beta) = n\pi + (-1)^n (x+iy)$$

put $n=0$

$$\Rightarrow \quad \sin^{-1}(\alpha+i\beta) = x+iy$$

whose real part $-\frac{\pi}{2} \le x \le \pi/2$

The General value of

$$\cos^{-1}(\alpha+i\beta) = 2n\pi \pm (x+iy)$$

Putting $n=0$

$$\cos^{-1}(\alpha+i\beta) = x+iy$$

its real part $0 \le x \le \pi$

The General value of.

$$\tan^{-1}(\alpha+i\beta) = n\pi + (x+iy)$$

Putting $n=0$

$$\tan^{-1}(\alpha+i\beta) = x+iy$$

Where $-\frac{\pi}{2} < x < \pi/2$

<u>Inverse Hyperbolic Functions</u> :→

If $\sinh z = w \Rightarrow z = \sinh^{-1} w$

$\cosh z = w \Rightarrow z = \cosh^{-1} w$

$\tanh z = w \Rightarrow z = \tanh^{-1} w$

Case I → Let $\sinh z = w$

where z and w both are complex Nos

Given by $z = x+iy$

$w = u+iv$.

$\Rightarrow z = \sinh^{-1} w$

Now $w = \sinh z = \dfrac{e^{z} - e^{-z}}{2}$

$= \dfrac{1}{2}\left[e^{z} - \dfrac{1}{e^{z}}\right]$

$\Rightarrow 2w = e^{z} - \dfrac{1}{e^{z}}$

$2we^{z} = (e^{z})^{2} - 1$

$\Rightarrow (e^{z})^{2} - 2we^{z} - 1 = 0$

$\Rightarrow e^{z} = \dfrac{2w \pm \sqrt{4w^{2}+4}}{2}$

$= w \pm \sqrt{w^{2}+1}$

$\therefore z = 2n\pi i + \log\left[w + \sqrt{w^{2}+1}\right]$

or $z = 2n\pi i + \log\left[w - \sqrt{w^{2}+1}\right]$

Since $w - \sqrt{w^{2}+1} = \dfrac{\left[w - \sqrt{w^{2}+1}\right]\left[w + \sqrt{w^{2}+1}\right]}{w + \sqrt{w^{2}+1}}$

$= \dfrac{w^{2} - w^{2} - 1}{w + \sqrt{w^{2}+1}}$

$= -\dfrac{1}{w + \sqrt{w^{2}+1}}$

$\therefore \log\left[w - \sqrt{w^{2}+1}\right] = \log(-1) - \log\left(w + \sqrt{w^{2}+1}\right)$

$= i\pi - \log\left[w + \sqrt{w^{2}+1}\right]$

Thus $z = 2n\pi i + \log\left[w + \sqrt{w^{2}+1}\right]$

or $z = (2n+1)\pi i - \log\left[w + \sqrt{w^{2}+1}\right]$

$= (2n+1)\pi i + (-1)^{2n+1} \log\left[w + \sqrt{w^{2}+1}\right]$

Let $w = \tanh z \Rightarrow z = \tanh^{-1} w$

Both the values of z can be included in the Expression

$\sinh^{-1} w = z = n\pi i + (-1)^n \log [w + \sqrt{w^2+1}]$

Above is the General value of $\sinh^{-1} w$

Putting $n=0$ we get the principal value of $z = \sinh^{-1} w$

$z = \log [w + \sqrt{w^2+1}]$

(ii) Let $w = \cosh z \Rightarrow z = \cosh^{-1} w$

Now $w = \cosh z = \dfrac{e^z + e^{-z}}{2}$

$\Rightarrow 2w = e^z + e^{-z} = e^z + \dfrac{1}{e^z}$

$\Rightarrow (e^z)^2 - 2we^z + 1 = 0$

$\Rightarrow e^z = \dfrac{2w \pm \sqrt{4w^2-4}}{2}$

$= w \pm \sqrt{w^2-1}$

Hence $z = \log [w \pm \sqrt{w^2-1}]$

Since $w - \sqrt{w^2-1} = \dfrac{w^2 - w^2 + 1}{w + \sqrt{w^2-1}}$

$= \dfrac{1}{w + \sqrt{w^2-1}}$

$\therefore \log [w - \sqrt{w^2-1}] = -\log [w + \sqrt{w^2-1}]$

then General value of $z = \cosh^{-1} w$

$= 2n\pi i \pm \log [w + \sqrt{w^2-1}]$

Principal value. Put $n=0$

$z = \cosh^{-1} w = \log [w + \sqrt{w^2-1}]$

Let $w = \tanh z \Rightarrow z = \tanh^{-1} w$

Now $w = \dfrac{e^{z} - e^{-z}}{e^{z} + e^{-z}} = \dfrac{e^{2z} - 1}{e^{2z} + 1}$

$$\Rightarrow \frac{1}{w} = \frac{e^{2z} + 1}{e^{2z} - 1}$$

using componendo and dividendo.

$$\frac{1+w}{1-w} = e^{2z}$$

$$\Rightarrow 2z = \log \frac{1+w}{1-w}.$$

$\therefore$ General value.

$$2z = 2n\pi i + \log \frac{1+w}{1-w}.$$

$$z = n\pi i + \frac{1}{2} \log \frac{1+w}{1-w}$$

putting $n = 0$ we get principal value.

$$z = \tanh^{-1} w = \frac{1}{2} \log \frac{1+w}{1-w}$$

Relation between inverse Hyperbolic Functions and inverse circular functions —

$$\left.\begin{aligned} \sinh^{-1} w &= -i \sin^{-1}(iw) \\ \cosh^{-1} w &= -i \cos^{-1}(iw) \\ \tanh^{-1} w &= -i \tan^{-1}(iw) \end{aligned}\right\} \text{Important.}$$